Exam Calculus 1

4 february 2013, 9.00-12.00.

This exam has 9 problems. Each problem is worth 1 point; more details can be found below. Write on each page your name and student number, and on the first page your seminar group. The use of annotations, books and calculators is not permitted in this examination. All answers must be supported by arguments and/or work. Success.

1. (a) Formulate the principle of mathematical induction.

   (b) Prove that if $n \geq 1$ is a positive integer, then
   $$3^{2n-1} + 1$$
   is divisible by 4.

2. Determine all complex numbers z satisfying
   $$\mathrm{e}^{\mathrm{i}z} = -\tfrac{1}{2}\sqrt{2}\,(1+\mathrm{i})$$

3. (a) The function $f(x)$ is defined on the interval $(-\infty, a)$. Give the precise definition of
   $$\lim_{x\to-\infty} f(x) = L$$

   (b) Prove, using this definition, that
   $$\lim_{x\to-\infty} \frac{1}{x^2} = 0$$

   (c) Idem,
   $$\lim_{x\to-\infty} \frac{\cos x}{x^2} = 0$$

4. Let f be a function that satisfies the following hypothesis:

   - f is continuous on the closed interval $[a, b]$
   - f is differentiable on the open interval (a, b)

   (a) Formulate the Mean Value Theorem.

   (b) Assume that there exists two numbers m en M such that $m \leq f'(x) \leq M$ for all $x \in (a, b)$ Prove, using the Mean Value Theorem, that
   $$m(b-a) \;\leq\; f(b) - f(a) \;\leq\; M(b-a)$$

   (c) Prove (using part (b)) that for all real number a en b:
   $$|\cos a - \cos b| \;\leq\; |a - b|$$

5. Let f be a differentiable function and $f(x) > 0$ (for all x).
Fill in the missing pieces:

$$\frac{d}{dx}\sqrt{f(x)} = \lim_{h\to 0} \frac{\sqrt{f(x+h)} - \sqrt{f(x)}}{h} = \cdots = \frac{f'(x)}{2\sqrt{f(x)}}$$

6. Evaluate

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$$

7. The function f is given by

$$f(x) = x \sin \frac{1}{x}$$

if $x \neq 0$ and $f(0) = 0$.

(a) Is f continuous at 0?

(b) Is f differentiable at 0?

8. (a) Evaluate

$$\int (\ln x)^2 \, dx$$

(b) Evaluate

$$\int_0^3 \sqrt{9 - x^2} \, dx$$

9. Find the solution $y(x)$ of the differential equation

$$y' + 3x^2 y = 6x^2$$

that satisfies $y(0) = 1$.

Maximum score for each question:

1a	0.5	2	1.0	3a	0.3	4a	0.3	5	1.0	6	1.0	7a	0.6	8a	0.5	9	1.0
b	0.5			b	0.4	b	0.4					b	0.4	b	0.5		
				c	0.3	c	0.3										